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OPTICAL INTERPRETATIONS IN HIGHER GEODESY.[1]

By WM. H. ROEVER, Washington University.

It is the object of this paper to show that certain curves which are defined in higher geodesy may be given simple optical interpretations. In order to be able to state what these interpretations are it will be necessary to define certain terms in both higher geodesy and optics.

§ 1. **Terms in Higher Geodesy.** Let us assume a set of rectangular axes O–x, y, z which are at rest with respect to the solid part of the rotating earth and such that the axis Oz is coincident with the earth's axis of rotation, the positive direction of Oz being from the celestial south pole to the celestial north pole, and the positive directions of Ox and Oy being such that $+Oy$ may be obtained from $+Ox$ by rotating $+Ox$ through 90° in the direction in which the earth rotates. Let us denote by P_0, (x_0, y_0, z_0) a point (on or near the earth's surface) which is at rest with respect to these axes. The field of force which determines weight, that is, the field in which a plumb-line is in equilibrium, is at rest with respect to these axes. The straight line which passes through P_0 and gives the direction of the force of this field at P_0, is defined as the *vertical at* P_0. The vertical at P_0 coincides with the string of a plumb-line,[2] the plumb-bob of which is situated at P_0. The *astronomical latitude at* P_0 is the complement ϕ of the angle which the vertical at P_0 (in the direction of the zenith) makes with the earth's axis (in the direction of the celestial north pole). The vertical at a general point P_0 does not intersect the earth's axis of rotation.

The *astronomical meridian plane at* P_0 is the plane which passes through the vertical at P_0 and is parallel to the earth's axis of rotation. The *astronomical longitude at* P_0 is the angle λ which the meridian plane at P_0 makes with a fixed plane through the axis of rotation (we here assume that the fixed plane is the plane zOx) measured from 0° to 360° in the direction in which the earth rotates.

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[1] Presented to the American Mathematical Society (Chicago Section), April 5, 1912.

[2] It is assumed that the string of the plumb-line is weightless and perfectly flexible, and that the plumb-bob is a heavy particle.

The *horizontal plane at* P_0 is the plane which passes through P_0 and is perpendicular to the vertical at P_0. The *north-and-south line at* P_0 is the line of intersection of the meridian and horizontal planes at P_0. The *east-and-west line at* P_0 is the line which passes through P_0 and is perpendicular to the meridian plane at P_0. It is evident that this line is the intersection of the horizontal plane at P_0 by the plane which passes through P_0 and is perpendicular to the earth's axis of rotation.

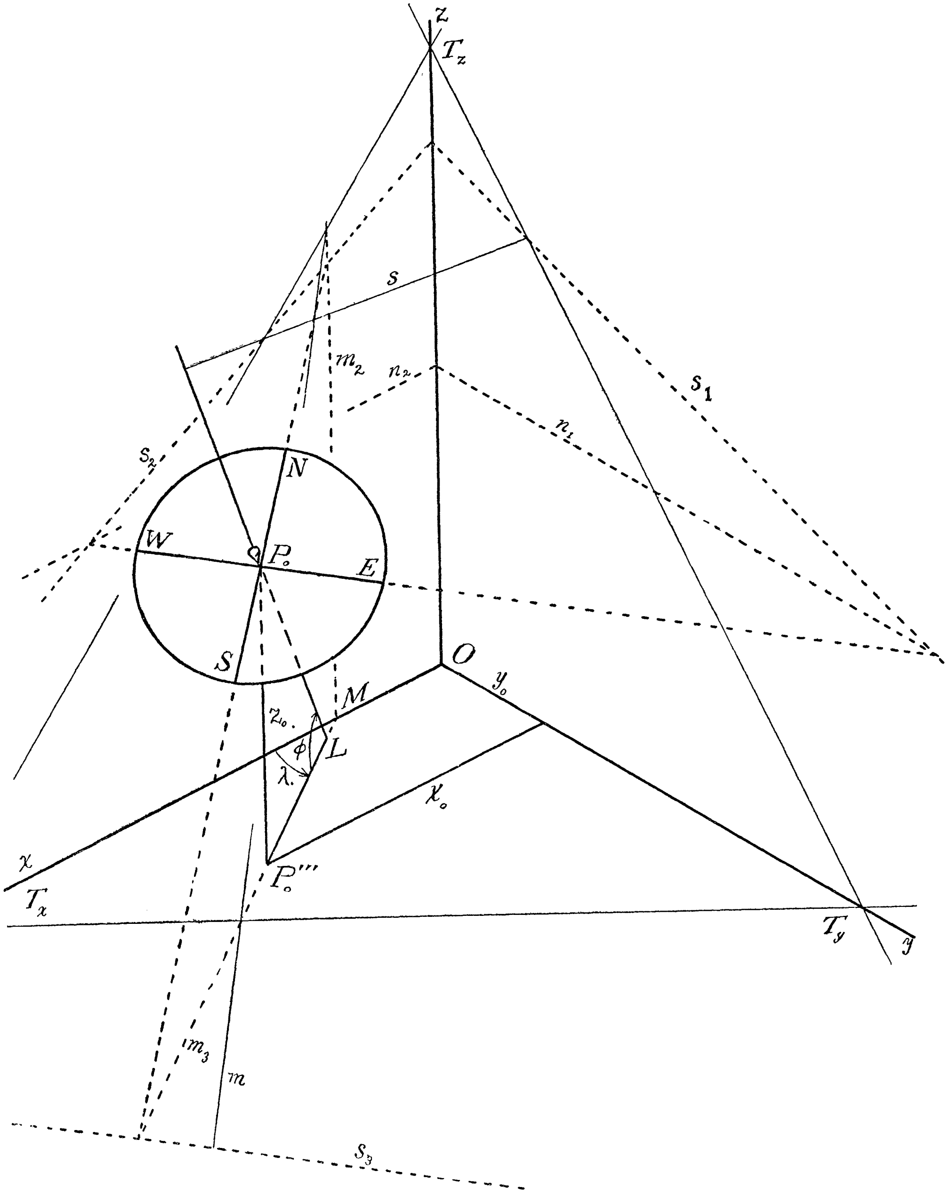

Fig. 1.

Thus in Fig. 1, in which the system of axes O-x, y, z is represented in orthographic projection,[1] is shown the point P_0, whose projection on the plane xOy is P_0''', the vertical P_0L, whose projection on the plane xOy is $P_0'''L$, the latitude $\phi = \measuredangle P_0'''LP_0$, the meridian plane LP_0P_0''' (also shown by its traces m_2, m_3), the longitude $\lambda = \measuredangle T_xMP_0'''$, the horizontal plane $NESW$ (also shown by its traces s_1, s_2, s_3), the north-and-south line NP_0S (coincident with the line connecting the intersection of s_2 and m_2 with that of s_3 and m_3), the east-and-west line EP_0W (coincident with the line connecting the intersections n_1s_1 and n_2s_2, where n_1 and n_2 are traces of the plane which passes through P_0 and is parallel to xOy).

A level surface is an equipotential surface of the field of force referred to above, *i. e.*, a surface, fixed with respect to the axes O-x, y, z, at each point of which the normal is the vertical. A *curve of constant astronomical latitude* is a locus of points, on a level surface, for which ϕ is constant. In particular, *the equator* is the locus for which $\phi = 0$. A *curve of constant astronomical longitude* is a locus of points, on a level surface, for which λ is constant.[2] The east-and-west line at a general point P_0 is *not* tangent to the curve of constant latitude which passes through P_0. A curve at each point of which the tangent is the east-and-west line we shall call an *east-and-west curve*.[3] It is not difficult to see that the east-and-west curves lie on level surfaces and also in planes which are perpendicular to the earth's axis of rotation.[4] Hence the one-parameter family of east-and-west curves, which lie on a particular level surface, are cut from this level surface by the one-parameter family of planes which are perpendicular to the earth's axis of rotation.

§ 2. **Terms in Optics.** If rays of light meet the surface of a body, the surface is said to be illuminated in the region where the rays meet it. The appearance of brightness of an illuminated surface depends upon the quantity of light which this surface sends to the retina of the eye. The quantity of light which an element of surface sends to the eye depends on the *intensity of illumination* of this surface element and on the reflecting power, or *albedo*, of the surface. For parallel ray illumination (such as that due to the sun or other distant sources of light) the *intensity of illumination* of a plane surface element, whose normal makes an angle ϵ with the direction of the rays of light, is given by the formula

$$B = L' \cos \epsilon,$$

where L' is the intensity of illumination of a plane surface which cuts the rays of light orthogonally.

Of the quantity of light which falls upon the surface of a body, a portion is absorbed, another portion is reflected and still another portion may be transmitted through the body. Only that portion which is reflected contributes to the brightness. The manner of the reflection depends upon the nature of the surface. A reflecting surface is said to be *perfectly smooth* when corresponding to each incident ray there is but a single reflected ray, the reflection occurring according to the law that the angles of incidence and reflection are equal and lie in the same plane: that is, *the normal to the surface at the point of reflection bisects the angle formed by the incident and reflected rays.* A reflecting surface is said to be *perfectly rough or dull* when it scatters the incident light which illuminates it in all directions and appears equally bright from all directions of view. The *brightness* of a surface element f is defined as the quantity of light which this surface element sends to the retina of the eye, divided by the area of the retinal image of f. For parallel ray illumination the *brightness* of a surface element is given by the formula

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[1] This figure is correctly drawn according to the rules of axonometry. $T_xT_yT_z$ is the axonometric triangle, m and s are the axonometric traces of the planes $[m_2, m_3]$ and $[s_2, s_3]$ respectively. Therefore in the plane of the drawing the line s is perpendicular to the line P_0L and the line m is perpendicular to the line s_3.

[2] The above definitions are practically the same as those given by Pezzetti, *Trattato di Geodesia Teoretica* (1905), § 5.

[3] Similarly, *north-and-south curves* might be defined. In general, such curves are different from the curves of constant longitude.

[4] This is a consequence of the fact, stated above, that the east-and-west line at P_0 is the intersection of the horizontal plane at P_0 by the plane which passes through P_0 and is perpendicular to the earth's axis. For, the horizontal plane at P_0 is the tangent plane at P_0 to the level surface which passes through P_0, and the plane through P_0 and perpendicular to the axis, is its own tangent plane, and in general the tangent to the curve of intersection of two surfaces is the intersection of the tangent planes to the two surfaces.

$$H = AL' \cos \epsilon,$$

where A is the albedo of the surface and L' and ϵ are quantities already defined. The brightness is independent of the direction from which the surface is viewed, provided the surface is perfectly dull.

For a surface which is not plane, the quantity H varies from point to point, but has constant values along certain curves of the surface. These curves, provided L' and A are constants, are the loci of points of the surface for which the angle ϵ, between the surface normal and the direction of the rays of light, has constant values. They are called *lines of constant intensity of illumination or curves of constant brightness.* The particular curve of this kind for which $\epsilon = 90°$, $\cos \epsilon = 0$, is called *the line of shade.*

Now let us consider a perfectly smooth reflecting surface as defined above. Such a surface has no brightness at any of its points, except at those isolated points from which emanate those unique rays of reflection which pass to the observer's eye. These points are very brilliant and hence are called *brilliant points.*[1]

Loci of Brilliant Points. The two ideal types of reflecting surface which are defined above may be regarded as extremes between which there lies another type which is worthy of consideration. For the perfectly smooth reflecting surface there corresponds to each incident ray but a *single reflected ray,* which emanates from the point where the incident ray pierces the surface, while for the perfectly dull reflecting surface there corresponds to each incident ray the *two-parameter family of reflected rays* which emanate from the point where the incident ray pierces the surface. For a type of reflecting surface which has already been considered by the author,[2] there corresponds to each incident ray the *one-parameter family of reflected rays* which emanate from the point P where the incident ray pierces the surface and form a cone of revolution whose vertex is P and whose axis is a tangent line to the surface at the point P. Such a surface may be realized[1] by scratching a perfectly smooth reflecting surface along the members of a one-parameter family of curves of the surface. The surface of a piece of metal can be thus scratched by holding a piece of emery cloth against it while it rotates in a lathe. Whereas perfectly smooth reflecting surfaces, which are illuminated, exhibit isolated *bright points* (defined above as brilliant points), surfaces of this type exhibit *bright curves*. See the photograph, Fig. 2. These curves are the loci of points which the author has defined as *brilliant points of curves*.[2]

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[1] The above definitions and formulas are those given by Weiner, *Lehrbuch der Darstellenden Geometrie,* in the chapters on Beleuchtungslehre, Vol. I, p. 390, Vol. II, p. 200.

[2] See the introduction to the author's paper, "Brilliant Points of Curves and Surfaces," *Trans. of the American Math. Society,* Vol. 9, No. 2, pp. 245–279.

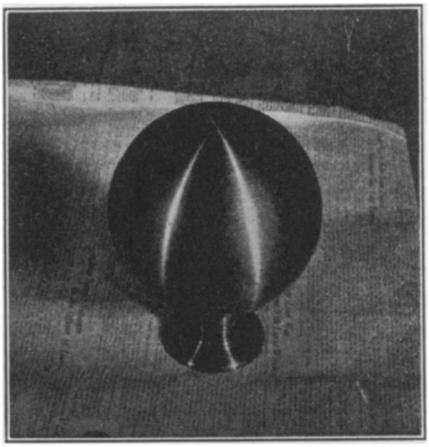

FIG. 2.

According to the law of reflection stated on page 72 and to the suggestion made in the last footnote, we have the following

Definitions. A point P is said to be a brilliant point of the surface t (or of the curve c, plane or twisted) with respect to the source (of light) P_1 and the recipient (an observer's eye) P_2 when the internal bisector of the angle P_1PP_2 is the normal to the surface t (or a normal to the curve c) at the point P. In particular P_1 and P_2 may be infinitely distant. As a consequence of these definitions the following theorem is immediately evident.

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[1] Of course, only approximately, as are also the other two types.

[2] In order to get the concept of " brilliant point of a curve " and to be able to see that there corresponds to each incident ray a cone of reflected rays, let us think of the scratches as forming ridges and furrows which are perfectly smooth, and furthermore let us think of each ridge or furrow as belonging to a tube-surface. As the cross-section of a tube-surface approaches zero, a brilliant point of the surface approaches a point of the curve which is approached by the vanishing tube-surface. This consideration suggests the definition of brilliant point of a curve which is here given. Those rays, reflected from a perfectly smooth tube-surface, which correspond to incident rays which lie in a plane, form a ruled surface. As the cross-section of the tube-surface approaches zero, the narrow band of incident rays approaches a single ray, and the ruled surface of reflected rays approaches one nappe of a cone of revolution, of which the vertex is the point Q where the incident ray meets the curve approached by the vanishing tube-surface, the axis is the tangent at Q to this limit curve and the angle is the same as that which the incident ray makes with the axis of this cone extended in the direction opposite to that on which this nappe lies.

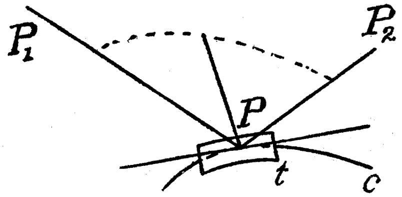

FIG. 3.

THEOREM. *The brilliant points of a surface (or a curve) are those of its points at which it is tangent to ellipsoids of revolution of which P_1 and P_2 are the foci. If, in particular, P_1 and P_2 are infinitely distant the confocal ellipsoids are replaced by parallel planes.*

If we denote by l_1, m_1, n_1 and l_2, m_2, n_2, the direction cosines of the directed lines PP_1 and PP_2 respectively, and by l, m, n numbers proportional to the direction cosines of the normal to the surface t (or the tangent line to the curve c) at the point P, then the condition that P should be a brilliant point, with respect to P_1 and P_2, of the surface is

$$(1) \qquad (l_1 - l_2)l + (m_1 - m_2)m + (n_1 - n_2)n = 0, \qquad \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0;$$

and of the curve c is[1]

$$(2) \qquad (l_1 + l_2)l + (m_1 + m_2)m + (n_1 + n_2)n = 0.$$

§ 3. **The Analogies.** Now that we have defined all of the necessary terms, we are able to state what the interpretations really are to which reference was made above. In order to do this let us think of a level surface of the earth's weight-field of force as being (1) a perfectly dull reflecting surface, (2) a reflecting surface of the type last defined in § 2 and scratched along the members of its one-parameter family of east-and-west curves.

We then have the following theorems:

THEOREM I. *The curves of constant astronomical latitude on a level surface are also the curves of constant brightness of this surface, regarded as a perfectly dull reflecting surface, when the source of illumination is an infinitely distant light in the direction of the earth's axis of rotation. In particular, the equator is the line of shade.*

THEOREM II. *The curves of constant astronomical longitude on a level surface are also the loci of the brilliant points of the one-parameter family of east-and-west curves of this surface, with respect to infinitely distant sources and recipients.*[2]

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[1] For more details see the author's paper already referred to.

[2] That is, if a level surface of the earth, supposed metallic, were scratched along its east-and-west curves, an observer on the moon, or any other distant body in a general direction, would see as a curve of light some curve of constant astronomical longitude, the curve being the assemblage of the images of the sun, which may be in any general direction, in the reflecting scratches.

The truth of theorem I follows immediately from the definitions of "brightness" and "astronomical latitude." The first of these quantities is proportional to cos ϵ and the second, ϕ, is equal to 90° − ϵ, since the rays of illumination are parallel to the earth's axis. Hence the locus of points on a level surface for which ϵ is constant, is in one case a curve of constant brightness and in the other, a curve of constant astronomical latitude.

Theorem II may be proved geometrically by means of the special case of the theorem stated in § 2. According to this special case, a point P of a curve c is a brilliant point of c with respect to the infinitely distant points P_1 and P_2, if c is tangent at P to one of the single infinity of planes which are perpendicular to the line b which bisects internally the angle P_1PP_2. If, in particular, the curve c is an east-and-west curve of a level surface, it lies in a plane perpendicular to the earth's axis of rotation a. Therefore, in order to find a brilliant point of an east-and-west curve c, it is only necessary to determine a point P of c at which the tangent is parallel to the line d in which the plane of c is cut by a plane perpendicular to the line b. The plane μ which is normal to an east-and-west curve c at a point P of c, and therefore also parallel to the earth's axis a, is the astronomical meridian plane of P. In the plane of each of the single infinity of east-and-west curves of a level surface there is a line d, a point P at which the tangent to the east-and-west curve c which lies in this plane is parallel to d, and a corresponding plane μ. All of the lines d are parallel (being perpendicular to both a and b) and hence all of the planes μ are parallel and consequently all of the points P have the same astronomical longitude. Therefore the locus of the brilliant points P is a curve of constant astronomical longitude.

In order to get an analytic proof of Theorem II, let us represent by ω the angular velocity of the earth's rotation, and by U the function $f(x, y, z)$ which is defined by the integral

$$U = \int_V \frac{dm}{\rho}, \qquad \rho = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2},$$

where dm represents that element of mass of the earth which is situated at the point whose coördinates, with respect to the axes O–x, y, z, are x_i, y_i, z_i, the integration being extended throughout the whole volume of the earth. Then the potential function of the earth's weight-field of force is[1]

$$W = U + \frac{\omega^2}{2}(x^2 + y^2).$$

The equations of the level surfaces are of the form

$$W(x, y, z) = C, \tag{1}$$

where C has constant values. For the particular level surface which passes through the point P_0, (x_0, y_0, z_0), $C = W(x_0, y_0, z_0)$. The direction cosines of the vertical at P_0 are proportional to[1]

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[1] See Pezzetti, *loc. cit.*, § 2.

$$W_x{}^{\circ}, \quad W_y{}^{\circ}, \quad W_z{}^{\circ}.$$

Since the direction cosines of the earth's axis (here coincident with Oz) are

$$0, \quad 0, \quad \pm 1,$$

the equation of the astronomical meridian plane at P_0 is

$$W_y{}^{\circ}(x - x_0) - W_x{}^{\circ}(y - y_0) = 0, \tag{2}$$

and hence the astronomical longitude λ at P_0 is given by the relation

$$\tan \lambda = \frac{W_y{}^{\circ}}{W_x{}^{\circ}}, \tag{3}$$

and the differential equations of the two-parameter family of east-and-west curves are[2]

$$\frac{dx}{W_y} = \frac{dy}{-W_x} = \frac{dz}{O}. \tag{4}$$

Since in our particular problem the source P_1 and the recipient P_2 are infinitely distant, the quantities $l_1, m_1, n_1, l_2, m_2, n_2$ are constants, as are also the quantities

$$\alpha = l_1 + l_2, \quad \beta = m_1 + m_2, \quad \gamma = n_1 + n_2,$$

which are proportional to the directional cosines of the bisector b defined in the geometric proof of theorem II. If now we replace l, m, n by $W_y, -W_x, O$ respectively, condition (2), § 2, takes the form

$$\alpha \cdot W_y - \beta \cdot W_x = 0 \quad \text{or} \quad \frac{W_y}{W_x} = \frac{\beta}{\alpha}. \tag{5}$$

This equation is the locus of the brilliant points, with respect to the infinitely distant points P_1 and P_2, of the two-parameter family of east-and-west curves (4). In its second form we recognize it also (by virtue of relation (3)) as the locus of points which have the same constant astronomical longitude $\lambda = \tan^{-1} (\beta/\alpha)$. Therefore the curve of intersection of this locus (5) with a level surface is both *a curve of constant astronomical longitude* on this level surface and *a locus of brilliant points*, with respect to the infinitely distant points P_1 and P_2, *of the one-parameter family of east-and-west curves* on this level surface. It is evident that we can choose the directions of the infinitely distant points P_1 and P_2 (in an infinite variety of ways) so as to make the expression $\lambda = \tan^{-1}(\beta/\alpha)$ for the astronomical longitude have any value we wish.[1]

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[1] The symbols W_x, W_y, W_z are used for the partial derivatives $\partial W/\partial x$, $\partial W/\partial y$, $\partial W/\partial z$ respectively, and the symbols $W_x{}^{\circ}$, $W_y{}^{\circ}$, $W_z{}^{\circ}$ stand for the values which these derivatives have at the point $x = x_0$, $y = y_0$, $z = z_0$.

[2] These differential equations may be written in the form: $W_x\,dx + W_y\,dy + W_z\,dz = 0$, $dz = 0$, from which it is evident that the solution is $W = C$, $z = k$, where C and k are constants. From this we see (what has already been stated in § 1) that the east-and-west curves are the intersections of the level surfaces with the planes perpendicular to the earth's axis.

From the form of the function W, it is evident that the level surfaces (1) are, in general, *not* surfaces of revolution. If, in particular, the distribution of the earth's matter were such that the function U assumed the form $f(v, z)$, where $v = \sqrt{x^2 + y^2}$, then the level surfaces would be surfaces of revolution, the east-and-west curves would be identical with the curves of constant astronomical latitude, and the curves of constant astronomical longitude would be the meridian curves[2] of these surfaces of revolution.

Hence Theorem II yields the following corollary.

COROLLARY. *The locus of the brilliant points, with respect to an infinitely distant source P_1 and an infinitely distant recipient P_2, of the one-parameter family of curves which are cut from a surface of revolution by planes perpendicular to the axis, is a meridian curve of this surface of revolution.*[3]

The photograph on page 73 shows two such curves (due to two sources of light) on a brass sphere which has been properly scratched with emery cloth while rotating in a lathe. In reality, the sources and the recipient (observer's eye) are not infinitely distant, but they are practically so.

A THEOREM IN THE MODERN PLANE GEOMETRY OF THE ABRIDGED NOTATION.

BY ROBERT E. BRUCE, Boston University.

NOTE BY THE EDITORS. The following paper is another contribution fulfilling the spirit of the editorial in the January issue. While the subject matter is strictly elementary, the methods are elegant and forceful and should furnish an inspirational study for those interested in modern geometry. Those who may wish to read an introduction to the abridged notation will find chapter IV of Salmon's *Conic Sections* helpful.

Introduction. In common with polar reciprocation, projection, and certain other methods of modern geometry, the abridged notation may be used either for the proof of a theorem presupposed to be true or for the discovery of new theorems the exact form of which is unsuspected until the proof is complete. For either purpose an identity in terms of the abridged notation is selected and any number of algebraic operations performed upon it leading to a final identity. The first identity corresponds to the hypothesis, the last to the conclusion, and the intervening transformations to the proof. If the form of the theorem is

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[1] It is also evident that the locus of the billiant points, with respect to a source P_1 and a recipient P_2, of any one-parameter family of curves on a surface σ, passes through the brilliant points, with respect to P_1 and P_2, of σ.

[2] A meridian curve of a surface of revolution is a curve which is cut from this surface by a plane which passes through the axis of rotation.

[3] If a surface of revolution were scratched along its meridian curves, the locus of the brilliant points would be a more complex curve. Such a locus has been considered by the author for the case of a sphere, in Vol XX, No. 10, p. 299 of this MONTHLY.